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DROPLET CONCENTRATION***

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Theoretical Expression for the Autoconversion Rate of the Cloud Droplet Number Concentration

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Abstract

Accurate parameterization of the autoconversion rate of the cloud droplet concentration (number autoconversion rate) is critical for evaluating aerosol indirect effects using climate models; existing parameterizations are empirical at best, however. A theoretical expression is presented in this contribution that analytically relates the number autoconversion rate to the liquid water content, droplet concentration and relative dispersion of the cloud droplet size distribution. The analytical expression is theoretically derived by generalizing the analytical formulation previously developed for the autoconversion rate of the cloud liquid water content (mass autoconversion rate). Further examination of the theoretical number and mass autoconversion rates reveals that the former is not linearly proportional to the latter as commonly assumed in existing parameterizations. The formulation forms a solid theoretical basis for developing multi-moment representation of the autoconversion process in atmospheric models in general.

1. Introduction

Microphysical processes of clouds and precipitation occur on scales smaller than grid sizes of most atmospheric models such as climate models, and need to be accurately parameterized. One such process is autoconversion whereby cloud droplets grow into embryonic raindrops. Since the late 1960s, great effort has been devoted to developing and improving parameterization of the autoconversion rate of the liquid water content (mass autoconversion rate hereafter) [Berry 1968; Kessler, 1969; Manton and Cotton, 1977; Liou and Ou, 1989; Baker, 1993; Liu and Daum, 2004; Liu et al. 2004, 2005, 2006a, b].

However, the autoconversion rate for the cloud droplet concentration (number autoconversion rate, hereafter) has received little attention. With growing recognition of the importance of droplet concentration and relative dispersion in cloud-related phenomena, along with advances in computer power, two-moment schemes for microphysical parameterizations that considers the mass and number autoconversion rates have found increasing applications [Beheng 1995; Khairoudinov and Kogan, 2000; Cohard and Pinty, 2000; Seifert and Beheng, 2001; Chen and Liu, 2000; Morrison et al. 2004; Zhang et al. 2007]. The pressing need for accurate parameterization of the number autoconversion rate has been reinforced by the increasing interest in cloud-climate interactions, and aerosol indirect effects [Boucher et al., 1995; Lohmann and Feichter, 2005; Rotstayn, 2000; Rotstayn and Liu, 2005].

Virtually all existing parameterizations for the number autoconversion rate have essentially followed an earlier study by Berry and Reinhardt [1974], assuming that the

number autoconversion rate is linearly proportional to the corresponding mass autoconversion rate, which itself is empirically obtained by curve-fitting numerical simulations from detailed microphysical models [Ziegler1985; Beheng 1994; Khairoutdinov and Kogan 2000; Seifert and Beheng 2001]. Therefore, existing parameterizations for the number autoconversion rate suffer from all the deficiencies of simulation-based expressions for the mass autoconversion rate [Liu and Daum, 2004; Liu et al. 2004, 2005, 2006a, b for details about the deficiencies], for example, lacking clear physics. It is desirable to derive a theoretical expression for the number autoconversion rate derived from first principles. Furthermore, the linear proportionality between the number and mass autoconversion rates commonly assumed in existing parameterizations is questionable as well and warrants rigorous examination.

In a series of publications [Liu and Daum, 2004; Liu et al. 2004, 2005, 2006a, b], we have presented a theoretical formulation for the mass autoconversion rate. The primary objective of this contribution is to generalize the formulation for the mass autoconversion rate to derive an analytical expression for the number autoconversion rate. The other objective is to combine the theoretical number and mass autoconversion rates to examine the validity of the common assumption of the linear proportionality between the number and mass autoconversion rates.

2. Generalized Expression for Autoconversion Rate

According to Liu et al. [2004, 2005, 2006b], the autoconversion rate for any bulk quantity Y can be generically written as

$$P_Y = P_{Y0} T_Y, \quad [1]$$

where P_Y is the autoconversion rate; P_{Y0} is the rate function describing the conversion rate after the onset of the autoconversion process, and $0 \leq T \leq 1$ is the threshold function

describing the transition behavior of the autoconversion process. The analytical expressions for P_{Y0} and T_Y will be derived below.

2. 1. General expression for rate function

Without loss of generality, we consider the quantity Y that is related to the δ -th power moment of the droplet size distribution such that

$$y = \alpha r^\delta, \quad (2a)$$

$$Y = \alpha \int r^\delta n(r) dr = \alpha N r_\delta^\delta, \quad (2b)$$

where r is the droplet radius, $n(r)$ is the droplet size distribution, N is the droplet concentration, α and δ are parameters indicative of the characteristics of Y and the order of the power moment, and r_δ is the δ -th mean radius of the droplet population. For example, the pair of $\alpha = (4/3\pi\rho_w)$ and $\delta = 3$ indicates that Y is the cloud liquid water content; the pair of $\alpha = 1$ and $\delta = 0$ indicates that Y represents the cloud droplet concentration N . Similar to the derivation of the mass rate function presented in Liu and Daum [2004], the rate function for Y is readily expressed as

$$P_{Y0} = \alpha \int n(r_1) dr_1 \int K(r_1, r_2) r_2^\delta n(r_2) dr_2, \quad (3)$$

where $r_{1,2}$ represent the radii of the collector and collected droplets, respectively, K is the collection kernel, and the integration is over all the droplets. Application to Eq. (3) of the Long collection kernel for $r_1 < 50 \mu\text{m}$, $K(r_1, r_2) = \kappa r_1^6$, and subsequent integration yields

$$P_{Y0} = a \kappa_2 N^2 r_6^6 r_\delta^\delta, \quad (4)$$

where the coefficient $\kappa_2 \approx 1.9 \times 10^{11}$ in $\text{cm}^{-3} \text{s}^{-1}$, r_1 is in cm, and the collection kernel K is in $\text{cm}^3 \text{s}^{-1}$ [Long, 1974]. Further application to Eq. (4) of the linear relationship between the mean radius of any order (r_p) and the mean volume radius (r_3), $r_p = \beta_p r_3$, leads to

$$P_{Y0} = \alpha \left(\frac{3}{4\pi\rho_w} \right)^{(6+\delta)/3} \kappa_2 \beta_6^6 \beta_\delta^\delta N^{-\delta/3} L^{(6+\delta)/3}, \quad (5)$$

where ρ_w is the water density, L is the liquid water content, and β_6 and β_δ are dimensionless parameters depending on the relative dispersion of the cloud droplet size distribution.

2.2. General expression for threshold function

As treated for the mass autoconversion rate [Liu *et al.*, 2005, 2006b], the threshold function for Y is given by

$$T_Y = \frac{P_Y}{P_{Y\infty}} = \left[\frac{\int_{r_c}^{\infty} r^6 n(r) dr}{\int_0^{\infty} r^6 n(r) dr} \right] \left[\frac{\int_{r_c}^{\infty} r^\delta n(r) dr}{\int_0^{\infty} r^\delta n(r) dr} \right], \quad (6)$$

where r_c is the critical radius beyond which the collection process starts to dominate the growth of cloud drops [Liu *et al.*, 2004]. Further evaluation of Eq. (6) requires specifying the mathematical form of the cloud droplet size distribution. It has been shown that cloud droplet size distributions are well described by the general Weibull droplet size distribution given by [Liu and Hallett, 1997; Liu and Daum 2000],

$$n(r) = \frac{qN}{r_0^q} r^{q-1} \exp \left[- \left(\frac{r}{r_0} \right)^q \right], \quad (7a)$$

where the parameter q is related to the relative dispersion (ε) of the cloud droplet size

$$\text{distribution through } \varepsilon = \left[\frac{2q\Gamma(2/q)}{\Gamma^2(1/q)} - 1 \right]^{1/2} \approx q^{-1}. \quad (7b)$$

Application of the general Weibull droplet size distribution to Eq. (6) leads to the following expressions describing the general threshold function:

$$T_Y = \gamma\left(\frac{6+q}{q}, x_{cq}\right) \gamma\left(\frac{\delta+q}{q}, x_{cq}\right), \quad (8a)$$

$$x_{cq} = \left(\frac{r_c}{r_0}\right)^q = \Gamma^{q/3} \left(\frac{3+q}{q}\right) x_c^{q/3}, \quad (8b)$$

$$x_c = 9.7 \times 10^{-17} N^{3/2} L^{-2}, \quad (8c)$$

where x_c is the ratio of the critical to mean masses, Γ and γ are the complete and incomplete gamma function, respectively [see Liu et al. 2004, 2005, 2006b for more discussions about x_c]. Combination of Eqs. (5) and (8) yields the general expression for the autoconversion rate of Y:

$$P_Y = \alpha \left(\frac{3}{4\pi\rho_w} \right)^{(6+\delta)/3} \kappa_2 \gamma\left(\frac{6+q}{q}, x_{cq}\right) \gamma\left(\frac{\delta+q}{q}, x_{cq}\right) \beta_6^6 \beta_\delta^\delta N^{-\delta/3} L^{(6+\delta)/3} \quad (9)$$

3. Number Autoconversion Rate

3.1. Theoretical expression

Equations (5), (8) and (9) suggests that the rate function, threshold function, and the autoconversion rate of any moment Y can be expressed as functions of liquid water content, droplet concentration and relative dispersion. And the general expressions are reduced to those previously derived for the mass autoconversion rate when $\alpha = (4/3\pi\rho_w)$ and $\delta = 3$. The number autoconversion rate is readily obtained by applying of $\alpha = 1$ and δ

=0 to the general expressions, i.e.

$$P_{N0} = \left(\frac{3}{4\pi\rho_w} \right)^2 \kappa_2 \frac{\Gamma\left(\frac{6+q}{q}\right)}{\Gamma^2\left(\frac{3+q}{q}\right)} L^2, \quad (10a)$$

$$T_N = \gamma\left(\frac{6+q}{q}, x_{cq}\right) \gamma(1, x_{cq}), \quad (10b)$$

$$P_N = \left(\frac{3}{4\pi\rho_w} \right)^2 \kappa_2 \frac{\Gamma\left(\frac{6+q}{q}, x_{cq}\right) \Gamma(1, x_{cq})}{\Gamma^2\left(\frac{3+q}{q}\right)} L^2. \quad (10c)$$

The derivation of the above equations uses the expression for β_p ,

$$\beta_p^p = \Gamma\left(\frac{p+q}{q}\right) \Gamma^{-2}\left(\frac{3+q}{q}\right).$$

3.2. Further examination

Equation (10c) coupled with Eqs. (8b, c, d) suggests that the number autoconversion rate depends on liquid water content, droplet concentration, and relative dispersion. Figure 1 illustrates the dependence of the number autoconversion rate on liquid water content calculated from Eq. (10c) at different values of droplet concentration [solid and dashed curves for $N = 50 \text{ cm}^{-3}$ and $N = 500 \text{ cm}^{-3}$; black and red curves for $\varepsilon = 0.33$ ($q = 3$) and $\varepsilon = 1$ ($q = 1$)]. Evidently, the number autoconversion rate generally increases with increasing liquid water content. The dependence can be characterized in two distinct regimes, which are dominated by the threshold function and rate function, respectively (threshold-dominated and rate-dominated hereafter). The number autoconversion rate increases faster in the threshold-dominated regime than that in the rate-dominated regime. A smaller relative dispersion (black curves) leads to a smaller

number autoconversion rate in both regimes, but the threshold-dominated regime exhibiting a steeper transition. The dependence of the number autoconversion rate on droplet concentration is more interesting. A smaller droplet concentration (dashed curves) gives rise to a larger number autoconversion rate in the threshold-dominated regime; but the dependence on droplet concentration diminishes in the rate-dominated regime where the curves for different droplet concentrations converge into a single curve. In short, except for its independence of droplet concentration in the rate-dominated regime, all the features of the number autoconversion rate are similar to those for the mass autoconversion rate reported previously [Liu and Daum, 2004; Liu et al. 2004, 2005, 2006a, b]. The feature that the number autoconversion rate should be described by two different functions is worth emphasizing, suggesting that existing parameterizations that have been often obtained by using a single function such as a power-law to fit detailed model results may distort the number autoconversion rate.

Furthermore, existing parameterizations for the number autoconversion rate assume that the number autoconversion rate is linearly proportional to the mass autoconversion rate. This assumption of linear proportionality is equivalent to assuming that all new “drizzle” drops have the same radius r^* [typical drop radius hereafter, Beheng 1994; Khairoutdinov and Kogan 2000; Seifert and Beheng 2001], i. e.,

$$P_N = \frac{3}{4\pi\rho_w r_*^3} P_L. \quad (11)$$

Differences between different parameterizations lie in the differences in their parameterizations for mass autoconversion rate, and especially in their choices of different values assigned to the typical drop radius. For example, $r^* = 32, 25,$ and $40 \mu\text{m}$ were chosen in Beheng [1994], Khairoutdinov and Kogan [2000], and Seifert and Beheng

[2001], respectively. Despite its widespread use, this linear proportionality assumption and the wide range of r_* values used by different authors remain unexamined.

The new theoretical expression for number autoconversion rate, coupled to that for the mass autoconversion rate previously presented in [Liu and Daum, 2004, Liu et al., 2004, 2005, 2006a, 2006b], allows a rigorous examination of this assumption of linear proportionality, or if r_* is a constant.

By relating the theoretical number autoconversion rate [Eq. (10c)] to the mass autoconversion rate presented previously [[Liu and Daum, 2004; 2006a], we obtain a theoretical expression for r_* ,

$$r_* = \left[\frac{\gamma\left(\frac{3+q}{q}, x_{cq}\right)}{\gamma(1, x_{cq})} \right]^{1/3} r_3. \quad (12)$$

Figure 2 shows some results calculated from Eq. (12). It is clear from this figure that instead of being a constant as commonly assumed in existing parameterizations, r_* varies substantially with droplet concentration, liquid water content, and relative dispersion. Furthermore, the dependency also features two distinct regimes: r_* first decreases with increasing mean volume radius, and then linearly increases with increasing mean volume radius beyond some point. Careful inspection of Eqs. (10) and (12) indicates that the first and second regimes are dominated by the threshold function and rate function, respectively. The dependence of r_* on liquid water content, droplet concentration and relative dispersion may be the reason for the various values of r_* used in existing parameterizations.

It is noted in passing that for the typical CDSD with $q = 3$ ($\epsilon = 0.33$), the theoretical expressions for number and mass autoconversion rates can be further simplified as

$$P_N = \left(\frac{3}{4\pi\rho_w} \right)^2 \kappa_2 (x_c^2 + 2x_c + 2) e^{-2x_c} L^2, \quad (13a)$$

$$P_L = \left(\frac{3}{4\pi\rho_w} \right)^2 \kappa_2 (x_c^2 + 2x_c + 2) (1 + x_c) e^{-2x_c} N^{-1} L^3 \quad (13b)$$

These two theoretical expressions should be readily applied to two-moment schemes for parameterizing the autoconversion process in atmospheric models.

4. Concluding Remarks

The analytical formulation previously derived for the mass autoconversion rate is first generalized to consider the rate of change of any moment of the cloud droplet size distribution caused by the autoconversion process. The general formulation is then applied to theoretically derive an analytical expression for the number autoconversion rate. It is shown that like the mass autoconversion rate, the number autoconversion rate depends on the liquid water content, droplet concentration and relative dispersion. The dependency is characterized by two distinct regimes: one is dominated by the threshold function and the other by the rate function. A single function such as a power-law as often used in existing parameterizations cannot fully describe such two-function behaviors. It is also shown that the number autoconversion rate is not linearly proportional to the mass autoconversion rate as commonly assumed in existing parameterizations.

It should be emphasized that although only the number autoconversion rate is examined in detail in this work, the extension to autoconversion rates for other quantities

such as radar reflectivity is straightforward using the general formulation. It is interesting to examine the impact of replacing existing parameterizations with the theoretical one on model results. The result is useful for differentiating precipitating from non-precipitating clouds using remote sensing techniques as well, which will be addressed in another paper.

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References

- Beheng, K. D., 1994: A parameterization of warm cloud microphysical conversion processes. *Atmos. Res.* **33**, 193-206.
- Berry, E. X., 1968: Modification of the warm rain process. Preprints 1st Nat. Conf. Weather Modification, Albany, Amer. Meteor. Soc., 81-88.
- Berry, E. X., and R. L. Reinhardt, 1974: An analysis of cloud drop growth by collection: Part II. Single initial distributions. *J. Atmos. Sci.*, **31**, 1825-1831.
- Boucher, O., H. L. Treut, and M. B. Baker, 1995: Precipitation and radiation modeling in a general circulation model: Introduction of cloud microphysical process. *J. Geophys. Res.* **D100**, 16395-16414.
- Cohard, J., and J. Pinty, 2000, A comprehensive two-moment warm microphysical bulk scheme. I: Description and tests, *Q. J. R. Meteor. Soc.*, **126**, 1815-1842.
- Kessler, E., 1969: On the Distribution and Continuity of Water Substance in Atmospheric Circulation. *Meteoro. Monogr.*, Vol.10, 84pp. American Meteorological Society, Mass, Boston.
- Khairoutdinov, M, and Y. Kogan, 2000: A new cloud physics parameterization in a large-eddy simulation model of marine stratocumulus. *Mon. Wea. Rev.*, **128**, 229-243.
- Liou, K. N., and S. C. Ou, 1989: The role of cloud microphysical processes in climate: An assessment from a one-dimensional perspective. *J. Geophys. Res.* **D94**, 8599-8607.
- Liu, Y., and P. H. Daum, 2004, Parameterization of the autoconversion process. Part I: Analytical formulation of the Kessler-type parameterizations, *J. Atmos. Sci.*, **61**, 1539-1548.

- Liu, Y. and J. Hallett, 1997, The “1/3” power-law between effective radius and liquid water content, *Q. J. R. Meteor. Soc.*, **123**, 1789-1795.
- Liu, Y., P. H. Daum, and R. McGraw, 2004, An analytical expression for predicting the critical radius in the autoconversion parameterization, *Geophys. Res. Lett.*, **31**, L06121, doi:10.1029/2003GL019117.
- Liu, Y., P. H. Daum, and R. McGraw, 2005: Size truncation effect, threshold behavior, and a new type of autoconversion parameterization. *Geophys. Res. Lett.*, **32**, L11811, doi:10.1029/2005GL022636.
- Liu, Y., P. H. Daum, and R. McGraw, 2006a, Parameterization of the autoconversion process. Part II: Generalization of Sundqvist-type parameterizations. *J. Atmos. Sci.*, **63**, 1103-1109.
- Liu, Y., P. H. Daum, R. McGraw, and M. Miller, 2006b, Generalized threshold function accounting for effect of relative dispersion on threshold behavior of autoconversion process. *Geophys. Res. Lett.*, **33**, L11804, doi:10.1029/2005GL025500.
- Lohmann, U., and J. Feichter, 2005, Global indirect aerosol effects: A review, *Atmos. Chem. Phys.*, **5**, 715-737..
- Long, A. B., 1974: Solutions to the droplet collection equation for polynomial kernels. *J. Atmos. Sci.*, **31**, 1040-1052.
- Manton, M. J., and W. R. Cotton, 1977: Formulation of approximate equations for modeling moist deep convection on the mesoscale. Atmos. Sci. Paper No. 266, Dept. of Atmos. Sci., Colorado State University.

- Milbrandt, J. A., and M. K. Yau, 2005, A multimoment bulk microphysics parameterization. Part II: A proposed three-moment closure and scheme description, *J. Atmos. Sci.*, **62**, 3065-3081.
- Morrison, H., J. A. Curry, and V. I. Khvorostyanov, 2005, A new double-moment microphysics parameterization for application in cloud and climate models. Part I: Description. *J. Atmos. Sci.*, **62**, 1665-1677.
- Rotstayn, L. D., 2000: On the "tuning" of the autoconversion parameterizations in climate models. *J. Geophys. Res.* **D105**, 15495-15507.
- Rotstayn, L. D., and Y. Liu, 2005, A smaller global estimate of the second indirect aerosol effect. *Geophys. Res. Lett.*, **32**, L05708, doi:10.1029/2004GL021922.
- Seifert, A., and K. D. Beheng, 2001: A double-moment parameterization for simulating autoconversion, accretion and self-collection. *Atmos. Res.*, **59-60**, 265-281.
- Ziegler, C. L., 1985, Retrieval of thermal and microphysical variables in observed convective storms. Part I: Model development and preliminary testing. *J. Atmos. Sci.*, **42**, 1487-1509.

Figure Captions

Figure 1. Dependence of the number autoconversion rate on liquid water content. The solid and dashed lines represent those for droplet concentration $N = 50$ and 500 cm^{-3} , respectively. The black and red colors represent those for relative dispersion $\varepsilon = 0.33$ ($q = 3$) and 1 ($q = 1$), respectively.

Figure 2. Dependence of the typical drop radius r_* on the mean-volume radius r_3 . The solid and dashed lines represent those for droplet concentration $N = 50$ and 500 cm^{-3} , respectively. The black and red colors represent those for relative dispersion $\varepsilon = 0.33$ ($q = 3$) and 1 ($q = 1$), respectively.



